

Fibrancy of Symplectic Homology In Cotangent Bundles

Thomas Kragh¹

Abstract: We describe how the result in [1] extends to prove the existence of a Serre type spectral sequence converging to the symplectic homology $SH_*(M)$ of an exact Sub-Liouville domain M in a cotangent bundle T^*N . We will define a notion of a fiber-wise symplectic homology $SH_*(M, q)$ for each $q \in N$, which will define a graded local coefficient system on N . The spectral sequence will then have page two isomorphic to homology of N with coefficients in this graded local system.

Contents

Contents	1
1 Introduction	1
2 Symplectic Homology of $M \subset T^*N$	2
3 Fiber-wise Symplectic Homology	5
4 Fibrancy and Sketch of Proof of Theorem 1	5
Bibliography	7

1 Introduction

Let N be any closed smooth manifold. Let $T^*N \xrightarrow{\pi} N$ denote the cotangent bundle with its canonical projection. We may define the Liouville 1-form λ_N on T^*N by

$$\lambda_{N(q,p)}(v) = p(\pi_*(v)), \quad q \in N, p \in T_q^*N, v \in T_{(q,p)}(T^*N).$$

It is well-known that $\omega = d\lambda_N$ is non-degenerate and hence a symplectic form on T^*N .

A (finite type) **Liouville domain** $M = (M, \lambda) = (M, \omega, \lambda)$ is a (compact) exact symplectic manifold ($\omega = d\lambda$) such that the symplectic dual vector field X of λ defined by

$$\omega(X, -) = \lambda$$

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points outwards at the boundary ∂M . We will only be concerned with compact Liouville domains, so without further warning all such will be of finite type. Note that this condition implies that λ pulled back to the boundary ∂M is a contact form.

Example 1.1. Given a Riemannian structure on N the disc bundle

$$D_R T^* N = \{(q, p) \in T^* N \mid \|p\| \leq R\}$$

of any radius $R > 0$ is a Liouville domain with the 1-form λ_N defined above.*

An **Exact Liouville embedding** $(M', \omega', \lambda') \subset (M, \omega, \lambda)$ is an embedding $M' \subset M$ of concurrent dimensions, such that $\omega' = \omega|_{M'}$ and there exists a smooth function

$$f: M' \rightarrow \mathbb{R}$$

such that $\lambda|_{M'} - \lambda' = df$.

Example 1.2. Any closed exact Lagrangian sub-manifold $L \subset D_R T^* N$ not intersecting the boundary of $D_R T^* N$ has by the Darboux-Weinstein theorem an extension to an exact Liouville embedding $D_\varepsilon T^* L \subset D_R T^* N$ for some small ε and some Riemannian structure on L . *

In Section 2 we will define symplectic homology $SH_*(M)$ of a Liouville domain M . In Section 3 We will describe a fiber-wise version of this. I.e. a symplectic homology depending on $q \in N$ denoted $SH_*(M, q)$. These will define a local coefficient system $SH_*(M, \bullet)$ on N and the following theorem may be thought of as a type of Serre spectral sequence for symplectic homology.

Theorem. *For any exact Liouville embedding $(M, \lambda) \subset (D_R T^* N, \lambda_N)$ there is a spectral sequence $(E_{*,*,*}^r, d_r)_{r \geq 1}$ such that*

- *it strongly converges to a filtered quotient of $SH_*(M)$ and*
- *the second page term $E_{n,m}^2$ is isomorphic to $H_n(N; SH_m(M, \bullet))$.*

As will be evident from the construction in Section 3 the local system $SH_*(M, \bullet)$ is trivial if $M \xrightarrow{\pi} N$ is not surjective. This implies that $SH_*(M)$ is trivial, and thus generalizes the fact that any exact Lagrangian $L \subset T^* N$ surjects to N (see [2]). Indeed, in the case of $D_\varepsilon T^* L$ the symplectic homology never vanishes because it is loop space homology (see [4]) with possibly twisted coefficients (see [1]). In [1] we also considered products in the case of $D_\varepsilon T^* L$ to get the strong result that exact Lagrangians are up to a finite covering space lift a homology equivalence. However, for simplicity we will not consider these extra structures here.

2 Symplectic Homology of $M \subset T^* N$

In this section we describe the symplectic homology of a Liouville domain M exact embedded in $T^* N$. The precise setup used in this section is important for understanding the construction in Section 3. One can define symplectic homology independent of the fact that M is in $T^* N$, but for the purpose of

proving Theorem 1 the following explicit construction is convenient. The fact that this actually defines symplectic homology is known as localization (see e.g. [3]).

We will need to consider Floer homology (infinite dimensional Morse homology) of restrictions of the action integral

$$A(\gamma) = \int_{\gamma} \lambda_N - H dt, \quad (1)$$

where $H: T^*N \rightarrow \mathbb{R}$ is a smooth Hamiltonian and $\gamma: I \rightarrow T^*N$ is a path. In this section, we will only consider closed loops $\gamma: S^1 \rightarrow T^*N$, and we denote the action restricted to these by A^Λ . It is well-known that the critical points of A^Λ are in 1-1 correspondences with closed time-1 periodic orbits of the Hamiltonian flow.

The symplectic homology will be defined as a limit of Floer homologies of A_s^Λ associated to a sequence of Hamiltonians H^s . We will define these rather explicitly. So let

$$W = (1 - \varepsilon, 1] \times \partial M \subset M \subset T^*N$$

denote a symplectic collar neighborhood of $\partial M \subset M$. I.e. the 1-form λ is given by $r\lambda|_{\partial M}$, where $\lambda|_{\partial M}$ denotes the pull back of λ to the boundary, and r is the coordinate in $(1 - \varepsilon, 1]$. We will only need to consider Hamiltonians H^s which are locally constant away from W and which only depend on r on W .

Choose any smooth function $f: (1 - \varepsilon, \infty) \rightarrow \mathbb{R}$ which is concave such that $f(r)$ tends to $-\infty$ as r tends to $1 - \varepsilon$ and $f(r)$ is 0 for $r \geq 1$. We then define a smooth family of smooth functions $f_s: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ for $s > 1$ such that

- $f_s(r) = f(r) + s$ when $f(r) \geq -s/2$,
- f_s'' has a unique 0 in $(1 - \varepsilon, 1)$, and
- $f_s(r) = 0$ when $0 \leq r < (1 - \varepsilon)$.

Notice that these imply that f_s is convex on the interval from $-\infty$ to the unique

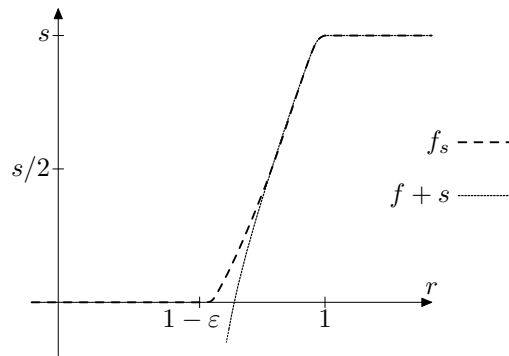


Figure 1: Functions f_s and $f + s$.

0 (in the second bullet point) and concave on the complement. As illustrated in Figure 1 this describes a controlled way to “cap of” $f + s$ making it smooth,

bounded, non-negative, and more we will need later. We then define the smooth family of smooth Hamiltonians H^s by

$$H^s(z) = \begin{cases} f_s(r) & z = (r, x) \in W \\ 0 & z \in M - W \\ s & z \in T^*N - M \end{cases}$$

By construction the Hamiltonian flow of H^s has only constant periodic orbits outside of W and hence the action on closed loops not in W only has critical values 0 and $-s$.

There is a geometric interpretation of the action of other periodic orbits. Indeed, since we only consider loops in this section and the embedding $M \subset T^*N$ is exact we may as well integrate over λ and not λ_N in Equation (1). Also, if $\gamma: S^1 \rightarrow \mathbb{R}$ is a Hamiltonian flow curve it has to have constant r -factor. Indeed, the Hamiltonian is preserved under Hamiltonian flow. Considering variations of this r -factor (knowing that γ is a critical point for A^Λ) yields the relation

$$\int_{\gamma} \lambda|_{\partial M} = f'_s(r), \quad (2)$$

and thus one calculates that

$$A_s^\Lambda(\gamma) = r f'_s(r) - f_s(r) = -(f_s(r) - r f'_s(r)),$$

which is minus the intersection of the 2. axis with the tangent of f_s at the point $(r, f_s(r))$.

Now pick $s_0 \gg 1$ large enough such that; the tangent of $f + s_0$ at the point where $f + s_0 = s_0/2$ intersects the 2. axis below zero. This and the assumptions on f_s implies that for $s > s_0$ we thus have; any tangent of f_s intersecting the 2. axis above 0 must be a tangent on the part where $f_s = f + s$ (and $f > -s/2$).

This means that any 1-periodic orbit with negative action must lie on an r -level for which $f_s(r) = f(r) + s$ (and also for r 's close to it). So when increasing s we simply increase f_s by the same value in a neighborhood. This implies that the 1-periodic orbit is unchanged and the action is simply translated down with the same speed as s is translated up. Thus the critical set of A_s^Λ has its critical values *below* zero pushed downwards with the same speed as s increases.

This means that for any $a < 0$ we in fact have continuation maps to higher and higher s and may in fact define the limit

$$SH_*(M) = \operatorname{colim}_{s \rightarrow \infty} FH_*^a(A_s^\Lambda), \quad (3)$$

where $FH_*^a(A_s^\Lambda)$ denotes Floer homology associated to critical values of A_s^Λ with critical value greater than a . Indeed, sliding a critical value below a corresponds to collapsing a generator in the Floer chain complex, which is always a chain map and which is why the continuation maps exist. Defining this colimit in the standard way requires picking a sequence of s 's tending to infinity and perturbing the associated actions (and some compactness proofs). However, to prove Theorem 1 we will not use standard Floer theory. We will instead use finite dimensional approximations. The advantages of this is addressed in Section 4. Note that the definition of this limit does not depend on $a < 0$ nor the choice of sequence s going to infinity. Indeed, all critical points with critical

values in $] - \infty, 0[$ for some s will eventually get their critical values translated down below any given value $a < 0$.

3 Fiber-wise Symplectic Homology

In this section we elaborate and extend the calculation of the action of 1-periodic orbits to open flow curves. We then use this to define a fiber-wise version of the symplectic homology.

Let $\gamma: I \rightarrow T^*N$ be any time-1 flow path for the Hamiltonian flow of H^s . The argument proving Equation (2) can be extended to this case using bump functions in the variable u . I.e. the equation is still valid. However, when using this to calculate A_s we get a correction term due to the fact that integrating λ and λ_N on open paths does not give the same result. Indeed, let $f: M \rightarrow \mathbb{R}$ be such that $df = \lambda_N - \lambda$ then

$$A_s(\gamma) = \int_{\gamma} \lambda_N - H^s dt = \int_{\gamma} \lambda - H^s dt + f(\gamma(1)) - f(\gamma(0)).$$

We conclude that the geometric interpretation using intersections of tangents of f_s with the 2. axis is still valid except we have to add the term $f(\gamma(1)) - f(\gamma(0))$.

Now fix a $q \in N$ and look at the action integrals A_s^q defined on paths γ starting and ending on the Lagrangian fiber T_q^*N . It is well-known that the critical points of A_s^q are time-1 Hamiltonian flow curves starting and ending on T_q^*N . This means that we can in fact conclude that if $a < \max_{z, z' \in M} |f(z) - f(z')|$ then the critical set of A_s^q below a slides down when s is increased. Indeed, as in the previous section the time-1 flow curves do not change but their action is translated along with s . So we may define

$$SH_*(M, q) = \operatorname{colim}_{s \rightarrow \infty} FH_*^a(A_s^q). \quad (4)$$

This is what we will consider the fiber-wise symplectic homology of M . By definition we see that if $M \rightarrow N$ does not surject there must be a fiber on which H^s is constantly equal to s . This implies that for this fiber the fiber-wise symplectic homology is trivial, and thus by theorem 1 all of them are trivial including the global symplectic homology $SH_*(M)$.

4 Fibrancy and Sketch of Proof of Theorem 1

The proof of Theorem 1 relies on the fact that the family of fiber-wise symplectic homologies actual looks and behaves like a Serre fibration with total space giving the total symplectic homology $SH_*(M)$. To prove this using homological algebra and infinite dimensional Floer theory seem very delicate and complicated - especially in light of transversality issues. So this is not the approach taken in [1]. Indeed, there we use finite dimensional approximations.

However, to see why the fiber-wise symplectic homologies form a local system and understand the construction it is convenient to discuss why a smooth path $\gamma: [0, l] \rightarrow N$ (assumed to be parametrized by arc-length) can induce a “parallel transport” in the fiber-wise homologies. So in the following we describe why such a path gives rise to a map between the fiber-wise symplectic homologies

$$SH_*(M, \gamma(0)) \rightarrow SH_*(M, \gamma(l)). \quad (5)$$

First consider any $s \gg 1$ fixed. Then for each $v \in [0, l]$ we have the fiber-wise action functional $A_s^{\gamma(v)}$ defined using the fiber $T_{\gamma(v)}^*N$ and one may consider the “graph” or **bifurcation diagram** of the critical set of each as a multi-valued function of v . Examples are illustrated in Figure 2. We know from the discussion

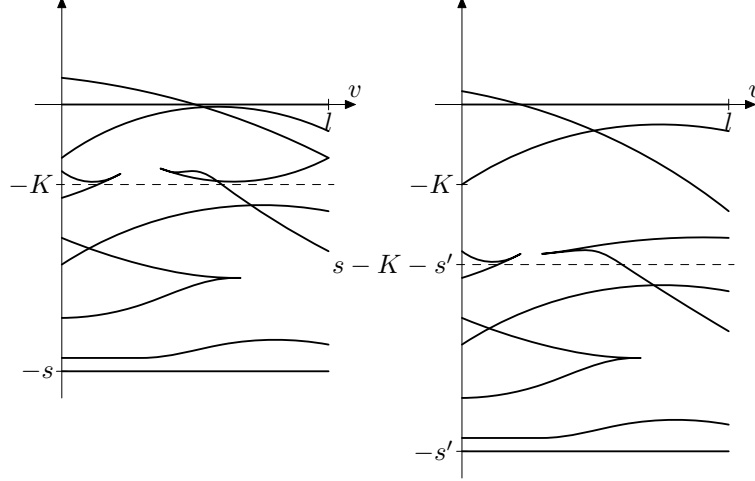


Figure 2: Bifurcation diagrams for same path but different s .

in the previous section that the dependence on s of this diagram is such that everything below some $-K$ is simply translated downwards with the same speed as s is translated upwards. Figure 2 shows two possible bifurcation diagrams for the same system and path γ , but for different s 's. The part above $-K$ can behave arbitrarily as s changes except there is always an upper bound. This made the colimits of Floer homologies in the previous sections well-defined.

The main point is that the slopes of the “graph” pieces in the bifurcation diagrams are bounded by a certain number. Indeed, Lemma 9.2 in [1] gives this bound to be 2 if $M \subset DT^*N$, which can always be arranged by scaling M at the very beginning. Note that the proof there depends on having a Hamiltonian which have a slight slope at infinity, but in this heuristical argument we don't care about the constant critical points outside of M since we simply assume s very large so that they do not contribute to the Floer homology (i.e. the critical value $-s$ is much smaller than any $a < -K$ chosen as the energy cut-off).

The effect of such a bound is that if we make s actually depend on v and grow faster than $2v$, say $s(v) = s_0 + 3v$, then the bifurcation diagram depending on v becomes slanted. Indeed, the slopes of any critical values with value less than $-K$ has to lie in the interval $[-5, -1] = [-2, 2] - 3$. The upshot is that whenever a critical value of $A_{s(v)}^{\gamma(v)}$ co-insides with a it must go down (as a function of v) with positive speed. We are thus again collapsing generators in the Floer chain complex and have maps similar to the continuation maps described in the previous sections. I.e. we get maps

$$FH_*^a(A_{s_0}^{\gamma(0)}) \rightarrow FH_*^a(A_{s_0+3l}^{\gamma(l)}),$$

which when taking the limit as s_0 tends to infinity defines the wanted map from Equation (5). Usual zig-zagging arguments can be employed to prove that

this is an isomorphism, but there are of course many things to check here, and proceeding using standard Floer theory seems very cumbersome.

The proof of Theorem 1 is carried out in [1] using the theory of Conley indices to produce spaces instead of homology theories. I.e. the spaces has the wanted homology. This makes it possible to define these fiber-wise symplectic homologies even when a from Equation (4) is not regular for a particular fiber T_q^*N and some s . Indeed, this is needed because there is no guarantee that we can find a sequence of s 's tending to infinity and being regular for all fibers simultaneously. In fact, in many cases this can be proven not to exist.

Working with spaces also has the advantage that it is possible to prove a Serre type fibration property in a more conventional manor, and thus lifting more than simply a path parametrized by arc length, but any compact family of smooth paths. It is important here to mention a technical but significant complication; when increasing s these finite dimensional approximations gets more and more complicated and suspensions of the Conley indices are introduced. This effectively means that the colimits analogous to those in Equation (3) and Equation (4) are not taken in the category of spaces, but in the category of spectra - and in the fiber-wise case even in a category of parametrized spectra over N . However, the notions of Serre fibrancy, taking fiber-wise homology, and taking global homology still exist. So we, indeed, get a Serre type spectral sequence as in Theorem 1. We should note, however, that in [1] only the case of $M = DT^*L$ is considered, but all the ideas generalize, and using the Hamiltonians defined here all the methods to get the spectral sequence immediately apply. However, to incorporate the products, which is very important in [1] one needs to construct the Hamiltonian family H^s more carefully.

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